Diploma Programme
Programme du diplôme
Programa del Diploma

Further mathematics
Higher level
Paper 2

Friday 18 May 2018 (morning)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 17]

The independent random variables X and Y are given by $X \sim \mathrm{N}\left(\mu_1, \sigma_1^2\right)$ and $Y \sim \mathrm{N}\left(\mu_2, \sigma_2^2\right)$.

(a) Write down the distribution of $aX + bY$ where $a, b \in \mathbb{R}$. [2]

(b) Two independent random variables X_1 and X_2 each have a normal distribution with a mean 3 and a variance 9. Four independent random variables Y_1, Y_2, Y_3, Y_4 each have a normal distribution with mean 2 and variance 25. Each of the variables Y_1, Y_2, Y_3, Y_4 is independent of each of the variables X_1, X_2. Find

(i) $\mathrm{P}\left(X_1 + Y_1 < 11\right)$;

(ii) $\mathrm{P}\left(3X_1 + 4Y_1 > 15\right)$;

(iii) $\mathrm{P}\left(X_1 + X_2 + Y_1 + Y_2 + Y_3 + Y_4 < 30\right)$. [10]

(c) Given that $\bar{X}$ and $\bar{Y}$ are the respective sample means, find $\mathrm{P}\left(\bar{X} > \bar{Y}\right)$. [5]

2. [Maximum mark: 17]

It is given that $(5x + y)\dfrac{\mathrm{d}y}{\mathrm{d}x} = (x + 5y)$ and that when $x = 0$, $y = 2$.

(a) Use Euler's method with step length 0.1 to find an approximate value of y when $x = 0.4$. [5]

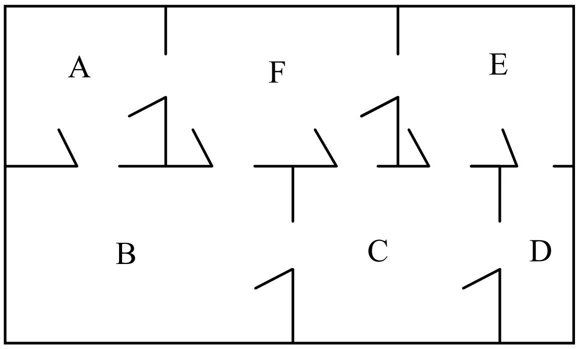
(b) (i) Show that $(5x + y)\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 1 - \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$.

(ii) Show that $(5x + y)\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} = -5\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)$.

(iii) Find the Maclaurin expansion for y up to and including the term in x^3. [12]

3. [Maximum mark: 17]

While on holiday Pauline visits the local museum. On the ground floor of the museum there are six rooms, A, B, C, D, E and F. The doorways between the rooms are indicated on the following floorplan.



(a) Draw a graph G to represent this floorplan where the rooms are represented by the vertices and an edge represents a door between two rooms. [2]

(b) (i) Explain why the graph G has an Eulerian trail but not an Eulerian circuit.

(ii) Explain the consequences of having an Eulerian trail but not an Eulerian circuit, for Pauline's visit to the ground floor of the museum. [4]

(c) (i) Write down a Hamiltonian cycle for the graph G.

(ii) Explain the consequences of having a Hamiltonian cycle for Pauline's visit to the ground floor of the museum. [3]

(This question continues on the following page)

(Question 3 continued)

There are 6 museums in 6 towns in the area where Pauline is on holiday. The 6 towns and the roads connecting them can be represented by a graph. Each vertex represents a town, each edge represents a road and the weight of each edge is the distance between the towns using that road. The information is shown in the adjacency table.

Vertices	U	V	W	X	Y	Z
U	-	11	10	7	11	12
V	11	-	5	13	4	6
W	10	5	-	15	10	10
X	7	13	15	-	9	15
Y	11	4	10	9	-	7
Z	12	6	10	15	7	-

Pauline wants to visit each town and needs to start and finish in the same town.

(d) Use the nearest-neighbour algorithm to determine a possible route and an upper bound for the length of her route starting in town Z. [2]

(e) By removing Z, use the deleted vertex algorithm to determine a lower bound for the length of her route. [6]

4. [Maximum mark: 14]

(a) Draw slope fields for the following cases for $-2 \le x \le 2$, $-2 \le y \le 2$

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$;

(ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x + 1$;

(iii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x - 1$. [6]

(b) Explain what isoclines tell you about the slope field in each of the following cases,

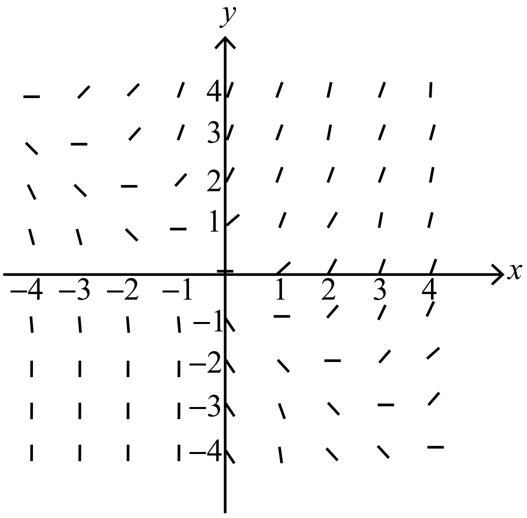
(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} =$ constant;

(ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)$. [2]

(This question continues on the following page)

(Question 4 continued)

(c) The slope field for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$ for $-4 \le x \le 4$, $-4 \le y \le 4$ is shown in the following diagram.



Explain why the slope field indicates that the only linear solution is $y = -x - 1$. [2]

(d) Given that all the isoclines from a slope field of a differential equation are straight lines through the origin, find two examples of the differential equation. [4]

5. [Maximum mark: 20]

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(a) Show that the area enclosed by the ellipse is πab. [9]

(b) The area enclosed by the ellipse is 8π and $b = 2$.

(i) Determine which coordinate axis the major axis of the ellipse lies along.

(ii) Hence find the eccentricity.

(iii) Find the coordinates of the foci.

(iv) Find the equations of the directrices. [8]

(c) The centre of another ellipse is now given as the point $(2, 1)$. The minor and major axes are of lengths 3 and 5 and are parallel to the x and y axes respectively. Find the equation of the ellipse. [3]

Turn over

6. [Maximum mark: 19]

The set of all integers from 0 to 99 inclusive is denoted by S. The binary operations $*$ and $\circ$ are defined on S by

$$a * b = [a + b + 20](\text{mod } 100)$$
$$a \circ b = [a + b - 20](\text{mod } 100).$$

(a) Find the identity element of S with respect to $*$. [3]

(b) Show that every element of S has an inverse with respect to $*$. [2]

(c) State which elements of S are self-inverse with respect to $*$. [2]

(d) Prove that the operation $\circ$ is not distributive over $*$. [5]

The equivalence relation R is defined by $aRb \Leftrightarrow \left(\sin\frac{\pi a}{5} = \sin\frac{\pi b}{5}\right)$.

(e) Determine the equivalence classes into which R partitions S, giving the first four elements of each class. [5]

(f) Find two elements in the same equivalence class which are inverses of each other with respect to $*$. [2]

7. [Maximum mark: 24]

(a) (i) In a triangle ABC, prove $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

(ii) Prove that the area of the triangle ABC is $\frac{1}{2}ab\sin C$.

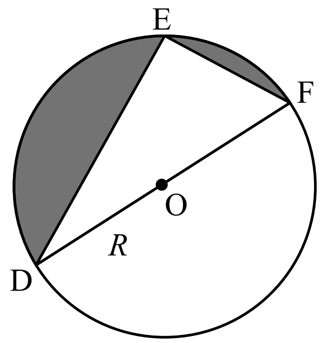
(iii) Given that R denotes the radius of the circumscribed circle prove that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

(iv) Hence show that the area of the triangle ABC is $\frac{abc}{4R}$. [10]

(This question continues on the following page)

(Question 7 continued)

(b) A new triangle DEF is positioned within a circle radius R such that DF is a diameter as shown in the following diagram.



(i) Find in terms of R, the two values of $(\mathrm{DE})^2$ such that the area of the shaded region is twice the area of the triangle DEF.

(ii) Using two diagrams, explain why there are two values of $(\mathrm{DE})^2$. [11]

(c) A parallelogram is positioned inside a circle such that all four vertices lie on the circle. Prove that it is a rectangle. [3]

8. [Maximum mark: 22]

The discrete random variable X follows a geometric distribution $\mathrm{Geo}(p)$ where

$$\mathrm{P}(X = x) = \begin{cases} pq^{x-1}, \text{ for } x = 1, 2... \\ 0, \text{ otherwise} \end{cases}.$$

(a) (i) Show that the probability generating function of X is given by

$$G(t) = \frac{pt}{1-qt}.$$

(ii) Deduce that $\mathrm{E}(X) = \frac{1}{p}$. [7]

(This question continues on the following page)

(Question 8 continued)

(b) Two friends A and B play a ball game with the following rules.

Each player starts with zero points. Player A serves first and then the players have alternate pairs of serves so that the service order is $\text{A}, \text{B}, \text{B}, \text{A}, \text{A}, \ldots$ When player A serves, the probability of her scoring 1 point is p_A and the probability of B scoring 1 point is q_A, where $q_A = 1 - p_A$.

When player B serves, the probability of her scoring 1 point is p_B and the probability of A scoring 1 point is q_B, where $q_B = 1 - p_B$.

Show that, after the first 6 serves, the probability that each player has 3 points is

$$\sum_{x=0}^{x=3} \binom{3}{x}^2 (p_A)^x (p_B)^x (q_A)^{3-x} (q_B)^{3-x}.$$ [5]

(c) After 6 serves the score is 3 points each. Play continues and the game ends when one player has scored two more points than the other player. Let N be the number of further serves required before the game ends. Given that $p_A = 0.7$ and $p_B = 0.6$ find $\text{P}(N = 2)$. [3]

(d) Let $M = \frac{1}{2}N$. Show that M has a geometric distribution and hence find the value of $\text{E}(N)$. [7]